Discussion on the Non-conservative Nature of the Induced Electric Field

Faraday's Principle and the Non-conservativeness of Magnetic Fields

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1. Preface: The Limitations of Kirchhoff's Theorems

Kirchhoff's voltage law (KVL) is a fundamental tool for analyzing electrical circuits.

However, when dealing with an induced electromotive force (EMF) arising from a time-varying magnetic field, the situation becomes nontrivial.

Charges move directionally without a conventional power source.

When integrating the electric field along the loop, one finds that：

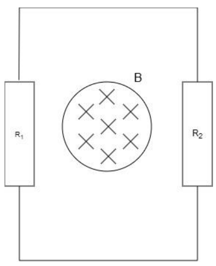
$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi_B}{dt} \neq 0$$

There is no doubt that this violates the conditions of Kirchhoff's principle: the electric field is a conservative field:

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0 \;\Rightarrow\; \nabla \times \mathbf{E} = 0$$

Therefore, we conclude that the induced electric field is a non-conservative field.

2. The circuit model of induced EMF



The magnetic flux can be calculated as:

$$\Phi_B(t) = \int_S \mathbf{B} \cdot d\mathbf{a}$$

According to Faraday's law of electromagnetic induction:

$$\oint_C \mathbf{E} \cdot \mathbf{dl} = -\frac{\mathbf{d\Phi_B}}{\mathbf{dt}}$$

When $\frac{d\Phi_B}{dt} \neq 0$, an induced electromotive force appears in the loop $\mathcal{E} = -\frac{d\Phi_B}{dt}$.

The charges are driven through these resistors, generating current:

$$I = \mathcal{E}/(R_1 + R_2)$$

If we directly apply the Kirchhoff's law, we can find that：

$$V_{R_1} + V_{R_2} + \mathcal{E} = 0$$

Although the numerical result coincides with the physical value, this equivalence arises only under specific approximations, not from the law's strict validity.

3. Why the KVL Result Remains Numerically Correct

(1). Neglecting Spatial Distribution:

For the current problem, under conditions of low frequency and uniform magnetic field variation, the entire conductor can be regarded as a one-dimensional device, and the changing magnetic flux can be replaced by an equivalent induced electromotive force:

$$\mathcal{E}_{ind} = -\frac{d\Phi_B}{dt}$$

(2). Neglecting Electromagnetic Radiation:

At low frequencies, wires do not radiate electromagnetic waves significantly. The energy provided by the changing magnetic field is almost entirely consumed in the form of electrical work or heat on the resistance.

Therefore, it can be idealized as a quasi-steady-state system with no radiative energy loss.

(3). Consistency in Total EMF:

Although the real electric field is continuously distributed and non-conservative within the conductor, the total loop integral value of it is the same as that of the equivalent voltage source.

Therefore, when we write in the circuit equation:

$$V_{R_1} + V_{R_2} + \mathcal{E}_{ind} = 0$$

In short, KVL remains numerically valid because we implicitly treat the distributed induced field as a lumped voltage source and neglect wave propagation effects.

4. Mathematical Proof of Non-conservativeness

Suppose the induced electric field E is conservative, then:`

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0$$

By Stokes' theorem, we have:

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = \iint_S (\nabla \times \mathbf{E}) \cdot d\mathbf{a}$$

Substitute Faraday's law:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Score:

$$0 = -\frac{d}{dt}\iint_S \mathbf{B} \cdot d\mathbf{a} = -\frac{d\Phi_B}{dt}$$

It is clearly in contradiction with the premise condition $\frac{d\Phi_B}{dt} \neq 0$, so the assumption does

not hold, and the induced electric field **E** is non-conservative.

5. Energy conservation perspective

The power of the electric field on the charge in the circuit is:

$$P = I \oint_C \mathbf{E} \cdot d\mathbf{l} = -I \frac{d\Phi_B}{dt}$$

From Poynting's theorem,

$$\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{S} = -\mathbf{J} \cdot \mathbf{E}$$

where:

$$u = \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{B^2}{\mu_0}\right), \quad \mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}$$

This indicates that:

The energy density **u** of the electromagnetic field varies with time;

Energy flow **S** transmits energy to the conductor;

The electric field does continuous work along a closed path.

Therefore, the induced electric field is a non-conservative energy transfer field:

It can absorb energy from a magnetic field and convert it into heat or mechanical work.

A conservative field does not do this.

6. Extension: Why the Magnetic Field Is Also Non-conservative

A static and unchanging magnetic field is not a conservative field either.

From Maxwell's equations,

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

There are no sources of magnetic fields (magnetic mono-poles do not exist).

As long as there is an electric current, $\nabla \times \mathbf{B} \neq 0$

That is to say, even if the magnetic field is stationary, it still has a curl and is still not a conservative field.

7. Conclusion: Conservative and Non-conservative

Through the analysis of Faraday's law, Stokes' theorem, and the energy conservation equation, we can clearly see that in the presence of a time-varying magnetic field, the induced electric field satisfies:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Therefore, its loop integral is not zero and it cannot be described by a single-valued potential function, which is a typical non-conservative field.

At the circuit level, although Kirchhoff's voltage law is no longer strictly valid, under low-frequency and quasi-static conditions, the induced electromotive force generated by the time-varying magnetic flux can be regarded as an equivalent distributed voltage source. Under this approximation, the calculation results of KVL still remain consistent with Maxwell's equations.

Furthermore, from the perspective of energy conservation, the induced electric field can continuously do work on the charge carriers in a closed loop, and the energy comes from the change in the magnetic field rather than the redistribution of static potential energy. This fact once again demonstrates that the induced electric field is a dynamic non-conservative field driven by external energy input.

Ultimately, we can draw a broader conclusion:

Whether it is the induced electric field generated by a time-varying magnetic field or the magnetic field produced by a steady current, neither satisfies the conditions of a conservative field. They reflect the dynamic unity of electromagnetic fields − energy no longer exists solely in static potential differences but flows and exchanges continuously in space in the form of fields.